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Applied Mathematics Letters 19 (2006) 741–746

Applied Mathematics Letters

www.elsevier.com/locate/aml

Multiplicity of solutions for critical singular problems

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Received 27 December 2004; received in revised form 19 October 2005; accepted 27 October 2005

Abstract

In this work we deal with the class of critical singular quasilinear elliptic problems in $\mathbb{R}^N$ of the form

$$-\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \alpha|x|^{-bq}|u|^{q-2}u + \beta|x|^{-dr}k|u|^{r-2}u \qquad x \in \mathbb{R}^N, \tag{P}$$

where $1 < p < N$, $a < N/p$, $a \le b < a+1$, α and β are positive parameters, $q = q(a,b) \equiv Np/[N - p(a+1-b)]$ and $d \in \mathbb{R}$. Moreover, $1 < r < p^* = Np/(N-p)$ and $0 \le k \in L^{q/(q-r)}_{r(d-b)}(\mathbb{R}^N)$.

Multiplicity results are established by combining a version of the concentration–compactness lemma due to Lions with the Krasnoselski genus and the symmetric mountain-pass theorem due to Rabinowitz.

Keywords: Degenerate quasilinear equation; p-Laplacian; Compactness–concentration; Variational methods

1. Introduction and main results

In this work we deal with the class of critical singular quasilinear elliptic problems in $\mathbb{R}^N$ of the form

$$-\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \alpha|x|^{-bq}|u|^{q-2}u + \beta|x|^{-dr}k|u|^{r-2}u \qquad x \in \mathbb{R}^N, \tag{P}$$

where $1 < p < N$, $a < N/p$, $a \le b < a+1$, α and β are positive parameters, $q = q(a,b) \equiv Np/[N - p(a+1-b)]$ and $d \in \mathbb{R}$. Moreover, $1 < r < p^* = Np/(N-p)$ and $0 \le k \in L^{q/(q-r)}_{r(d-b)}(\mathbb{R}^N)$, where

$$L^q_b(\mathbb{R}^N) \equiv \left\{ u : \mathbb{R}^N \to \mathbb{R} : |u|^q_{L^q_b(\mathbb{R}^N)} \equiv ||x|^{-b}u|^q_q = \int_{\mathbb{R}^N} |x|^{-bq}|u|^q \mathrm{d}x < \infty \right\}.$$

The study of this type of equation is motivated by its various applications, for instance, in fluid mechanics, in Newtonian fluids, in flow through porous media, and in glaciology. See [1].

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2005.10.004

We look for solutions of problem (P) in the Sobolev space $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ defined as the completion of the space $C_0^\infty(\mathbb{R}^N)$ endowed with the norm

$$\|u\| \equiv \left[\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p \mathrm{d}x\right]^{1/p}.$$

After the pioneering paper by Brézis and Nirenberg [2], who studied the case $p = 2$, $k = 1$, and $a = b = d = 0$, many authors studied this kind of problem in a bounded domain or in $\mathbb{R}^N$, mainly when the operator involved is the Laplacian or the p-Laplacian with convex nonlinearities. For instance, see [3–7] and references therein. For the singular problem in bounded domains, we refer the reader to [8,9] and related papers in their references.

Ambrosetti et al. in [10] treated some problems involving concave and convex nonlinearities in a bounded domain for the case $p = 2$, $a = b = d = 0$. For unbounded domains we cite [11,12].

Problem (P) involving the p-Laplacian operator with singular nonlinearities ($a = d = 0$) in bounded domains was studied, e.g., by Huang in [13], and Ghoussoub and Yuan in [9].

Recently, Chen and Li in [8] extended some results in [9] and also in [13] for problem (P) with $a = d = 0$ in $\mathbb{R}^N$. Their proof is constructed by combining some arguments used in [6,9] and [13] together with a version of the concentration–compactness principle due to Lions proved by Smets in [14]. This compactness result, for the singular operator with $p = 2$, was generalized by Wang and Willem [15], and, for the case $p \neq 2$, by Assunção et al. in [16]. See also Tan and Yang [17]. We recall that the techniques used in [8] in the case $a = 0$ do not hold immediately when $a \neq 0$.

By using this principle, we extend some results in [8,9] treating a problem involving singularities not only in the nonlinearities but also in the operator.

A well known result by Caffarelli et al. [18] guarantees that the Euler–Lagrange functional $I : \mathcal{D}_a^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) \equiv \frac{1}{p}\int_{\mathbb{R}^N} |x|^{-pa}|\nabla u|^p \mathrm{d}x - \frac{\alpha}{q}\int_{\mathbb{R}^N} |x|^{-bq}|u|^q \mathrm{d}x - \frac{\beta}{r}\int_{\mathbb{R}^N} |x|^{-dr} k|u|^r \mathrm{d}x \tag{1}$$

is well defined. Actually, $I \in C^1(\mathcal{D}_a^{1,p}(\mathbb{R}^N), \mathbb{R})$ and a weak solution u of problem (P) is precisely a critical point of the functional I, that is, $I'(u) = 0$ where

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} |x|^{-pa}|\nabla u|^{p-2}\nabla u \nabla v\, \mathrm{d}x - \alpha\int_{\mathbb{R}^N} |x|^{-bq}|u|^{q-2}uv\mathrm{d}x - \beta\int_{\mathbb{R}^N} |x|^{-dr}k|u|^{r-2}uv\, \mathrm{d}x.$$

Let the Lagrange's multiplier be given by

$$S(a,b) \equiv \inf_{\substack{u\in\mathcal{D}_a^{1,p}(\mathbb{R}^N)\\ u\not\equiv 0}} \frac{\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p \mathrm{d}x}{\left[\int_{\mathbb{R}^N} |x|^{-bq}|u|^q \mathrm{d}x\right]^{p/q}}.$$

Using the Caffarelli, Kohn and Nirenberg's inequality we can guarantee that $S(a,b)$ is a positive constant.

Our main results are the following.

Theorem 1. *Let $p < r < p^*$, $1 < p = q$ (that is, $b = a + 1$) and $a < (N - p)/p$ be given. Suppose that the Lebesgue measure of the set $\{x \in \mathbb{R}^N \mid k > 0\}$ is positive. Then for every $0 < \alpha < S(a, a + 1)$ and for every $\beta > 0$, problem (P) has infinitely many solutions whose corresponding energies are unbounded.*

Theorem 2. *Let $1 < r < p$, $1 < p < N$, q and $a < (N - p)/p$ be given. Suppose that the Lebesgue measure of the set $\{x \in \mathbb{R}^N \mid k > 0\}$ is positive. Then*

1. *For every $\alpha > 0$, there exists $B > 0$ such that if $0 < \beta < B$, then problem (P) has a sequence of solutions $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ with $I(u_n) < 0$ and $\lim_{n\to\infty} I(u_n) = 0$.*
2. *For every $\beta > 0$, there exists $A > 0$ such that if $0 < \alpha < A$, then problem (P) has a sequence of solutions $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ with $I(u_n) < 0$ and $\lim_{n\to\infty} I(u_n) = 0$.*

2. Auxiliary lemmas

The proofs of our results follow closely the approach used in [6] (see also in [8,13]). To find the critical points of the functional I, we have to establish conditions under which it verifies the Palais–Smale criterion $(PS)_c$, that is, conditions under which a sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in $(D_a^{1,p}(\mathbb{R}^N))^*$ has a convergent subsequence. This is done thanks to the concentration–compactness principle below, in whose statement we denote by $\mathcal{M}(\mathbb{R}^N)$ the space of positive, bounded measures in $\mathbb{R}^N$.

Lemma 3. *Let $N > p$, $0 \le a < (N-p)/p$, $a \le b \le a+1$, and q be given. Let $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ be a sequence such that the following convergences hold:*

1. $u_n \rightharpoonup u$ *weakly in* $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$.
2. $||x|^{-a}\nabla(u_n - u)|^p \rightharpoonup \mu$ *weakly in* $\mathcal{M}(\mathbb{R}^N)$.
3. $||x|^{-b}(u_n - u)|^q \rightharpoonup \nu$ *weakly in* $\mathcal{M}(\mathbb{R}^N)$.
4. $u_n \to u$ *a.e. in* $\mathbb{R}^N$.

We also define the measures of concentration at infinity

$$\nu_\infty \equiv \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\ge R} |x|^{-bq}|u_n|^q \mathrm{d}x,$$

$$\mu_\infty \equiv \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\ge R} |x|^{-ap}|\nabla u|^p \mathrm{d}x.$$

Then there exists an at most denumerable set of indexes J, such that

$$\nu = |x|^{-bq}|u|^q + \sum_{j\in J} \nu(\{x_j\}) > 0, \tag{2}$$

$$\mu \le |x|^{-ap}|\nabla u|^p + \sum_{j\in J} S(a,b)[\nu(\{x_j\})]^{p/q}, \tag{3}$$

$$[\nu(\{x_j\})]^{p/q} \le [S(a,b)]^{-1}\mu(\{x_j\}), \tag{4}$$

$$\nu_\infty^{p/q} \le [S(a,b)]^{-1}\mu_\infty. \tag{5}$$

Proof. The proof is similar to that of Lemma 4.1 in [16]; just consider adequately chosen functions $h \in C_o^\infty(\mathbb{R}^N)$, such that they are concentrated at each singularity. See also [15, Lemma 3] and [14, Lemma 2]. □

In what follows we study some properties of compactness of the functional I defined in (1). We begin by mentioning the boundedness of the Palais–Smale sequences, whose proof is standard.

Next we treat the existence of convergent subsequences in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$, which is the main step of the proof of our results.

Lemma 4. *Suppose that $1 < r < p$ and $a \le b < a+1$. Let $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ be a Palais–Smale sequence for the functional I at the level $c \in \mathbb{R}$. Then we have:*

1. *For every $\alpha > 0$, there exist $B > 0$ such that, if $0 < \beta < B$, then the sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ has a convergent subsequence in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$.*
2. *For every $\beta > 0$, there exists $A > 0$ such that, if $0 < \alpha < A$, then the sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ has a convergent subsequence in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$.*

Proof. We know that the sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ is bounded. Hence, passing to a subsequence, still denoted in the same way, we can suppose that the four conditions of Lemma 3 hold. Let $x_j \in \mathbb{R}^N$ be a singular point for the measures μ and ν; then, using an argument similar to that of Chen and Li [8], we can prove that $\nu(\{x_j\}) = 0$ and $\nu_\infty = 0$.

The following claim implies the lemma.

Claim 1. *The sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ converges strongly to $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N)$.*

Indeed, applying [19, Lemme 2.1] (see also [9]) for $p \geq 2$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} C|x|^{-ap}|\nabla u_n - \nabla u|^p \mathrm{d}x &\leq \int_{\mathbb{R}^N} |x|^{-ap}\langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u\rangle_e \,\mathrm{d}x \\ &= \langle I'(u_n) - I'(u), u_n - u\rangle + \alpha \int_{\mathbb{R}^N} |x|^{-bq}[|u_n|^{q-2}u_n - |u|^{q-2}u](u_n - u)\,\mathrm{d}x \\ &\quad + \beta \int_{\mathbb{R}^N} |x|^{-dr}k[|u_n|^{r-2}u_n - |u|^{r-2}u](u_n - u)\,\mathrm{d}x. \end{aligned} \tag{6}$$

Since $I'(u_n)$ tends to zero and by the weak continuity of the functional $F : \mathcal{D}_a^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$F(u) \equiv \int_{\mathbb{R}^N} |x|^{-dr}k|u|^r \mathrm{d}x,$$

it follows that the first and third terms in (6) tend to zero. See [8,13] for details.

For the second term in (6), using Hölder's inequality we have

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^{-bq}[|u_n|^{q-2}u_n - |u|^{q-2}u](u_n - u)\,\mathrm{d}x &\leq \left[\int_{\mathbb{R}^N} |x|^{-bq}|u_n|^q \mathrm{d}x\right]^{(q-1)/q} \left[\int_{\mathbb{R}^N} |x|^{-bq}|u_n - u|^q \mathrm{d}x\right]^{1/q} \\ &\quad + \left[\int_{\mathbb{R}^N} |x|^{-bq}|u|^q \mathrm{d}x\right]^{(q-1)/q} \left[\int_{\mathbb{R}^N} |x|^{-bq}|u_n - u|^q \mathrm{d}x\right]^{1/q}.\end{aligned}$$

Using the Brézis–Lieb Lemma and Lemma 3, when $\nu_j = 0$ and $\nu_\infty = 0$ we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-bq}|u_n - u|^q \mathrm{d}x = 0. \tag{7}$$

Therefore, this term also tends to zero. This concludes the proof of Claim 1 for the case $p \geq 2$. The case $1 < p < 2$ is similar. □

3. Case $1 < r < p$

Let $1 < r < p$ and $\alpha, \beta > 0$ be given. By Hölder's and Caffarelli, Kohn and Nirenberg's inequalities, we have

$$I(u) \geq \frac{1}{p}\|u\|^p - \alpha C_1\|u\|^q - \beta C_2\|u\|^r.$$

Following Garcia and Peral in [6] (see also [8,13]), we define the function

$$Q(t) \equiv \frac{1}{p}t^p - \alpha C_1 t^q - \beta C_2 t^r.$$

Hence, given $\beta > 0$, there exists $A > 0$ small enough such that, for every $0 < \alpha < A$, there exist $0 < T_0 < T_1$ verifying the following inequalities:

1. $Q(t) < 0$ for $0 < t < T_0$.
2. $Q(t) > 0$ for $T_0 < t < T_1$.
3. $Q(t) < 0$ for $t > T_1$.

Similarly, given $\alpha > 0$, there exists $B > 0$ small enough such that, for every $0 < \beta < B$, there exist $0 < T_0 < T_1$ verifying inequalities 1, 2, and 3 above. Clearly, $Q(T_0) = Q(T_1) = 0$ in both cases.

Let $\tau : \mathbb{R}^+ \to [0, 1]$ be a nonincreasing, differentiable function of class C^∞, such that $\tau(t) = 1$ for every $t \leq T_0$ and $\tau(t) = 0$ for every $t \geq T_1$. We define $\psi(u) \equiv \tau(\|u\|)$ and consider the truncated functional $\tilde{I} : \mathcal{D}_a^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\tilde{I}(u) \equiv \frac{1}{p}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p \mathrm{d}x - \frac{\alpha}{q}\psi(u)\int_{\mathbb{R}^N} |x|^{-bq}|u|^q \mathrm{d}x - \frac{\beta}{r}\int_{\mathbb{R}^N} |x|^{-dr}k|u|^r \mathrm{d}x. \tag{8}$$

Note that $\tilde{I} \in C^1(\mathcal{D}_a^{1,p}(\mathbb{R}^N), \mathbb{R})$ and is bounded from below. Moreover, if $\|u\| \geq T_0$, then $\tilde{I}(u) \geq 0$; hence, if $\tilde{I}(u) < 0$, then $\|u\| < T_0$ and $\tilde{I}(u) = I(u)$.

Now we will construct a minimax type sequence of appropriate negative critical values for the truncated functional $\tilde{I}$.

We begin by establishing the notation $K_c \equiv \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \mid \tilde{I}(u) = c \text{ and } \tilde{I}'(u) = 0\}$ and $\tilde{I}^c \equiv \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \mid \tilde{I}(u) \leq c\}$ for $c < 0$.

Remark 3.1. If α and β verify the conditions of Lemma 4, then it follows that every Palais–Smale sequence has a convergent subsequence. Hence K_c is compact and $\gamma(K_c)$ is well defined, where γ denotes Krasnoselski's genus. For the definition and properties of the genus, we refer the reader to [20, Chapter 7].

As in [8,6,13], using [20, Chapter 9 and Appendix A], and for every $m \in \mathbb{N}$, there exists $\varepsilon_m < 0$ such that

1. $\gamma(\tilde{I}^{\varepsilon_m}) \geq m$.
2. $c_m \equiv \inf_{A\in\Gamma_m} \sup_{u\in A} \tilde{I}(u)$, where $\Gamma_m \equiv \{A \in \Sigma \mid \gamma(A) \geq m\}$ and $\Sigma \equiv \{A \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N) \setminus \{0\} \mid A$ is closed and symmetric with respect to the origin$\}$.
3. $c_m \leq c_{m+1}$ and $-\infty < c_m \leq \varepsilon_m < 0$ for every $m \in \mathbb{N}$.
4. c_m is a critical value for the truncated functional $\tilde{I}$ and $\lim_{m\to\infty} c_m = 0$.

Using the above properties we conclude the proof of Theorem 2. □

4. Case $p < r < p^*$

We have the following compactness lemma.

Lemma 5. *Let $0 < \alpha < S(a, a+1)$ and $p < r < p^*$ be given. Let $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ be a Palais–Smale $(PS)_c$ sequence. Then for every $\beta > 0$ and for every $c \in \mathbb{R}$ the sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ has a convergent subsequence.*

Proof. Since the sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ is bounded, we can suppose that $u_n \rightharpoonup u$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$. Using Lemma 3 and the weak continuity of the functional F, we obtain $\mu(\{x_j\}) \leq \alpha\nu(\{x_j\})$ or $\nu(\{x_j\}) = 0$, where $x_j \in \mathbb{R}^N$ is a singular point for the measures μ and ν.

By Lemma 3, if $\nu(\{x_j\}) > 0$, then

$$\alpha\nu(\{x_j\}) \geq \mu(\{x_j\}) \geq S(a, a+1)\nu(\{x_j\})$$

which implies that $\alpha \geq S(a, a+1)$, a contradiction. Therefore, $\nu(\{x_j\}) = 0$. Similarly, $\nu_\infty = 0$.

Repeating the argument made in the proof of Lemma 4, more exactly the Claim 1, we have $u_n \to u$ in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ and this concludes the proof of the lemma. □

Proof of Theorem 1. The functional I verifies the Palais–Smale condition $(PS)_c$ by Lemma 5. It is standard to prove the following statements.

1. There exist constants $\rho > 0$ and $\delta > 0$ such that $\inf_{Z\cap\partial B_\rho} I \geq \delta$.
2. Let $X_m \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ be a m-dimensional subspace. Then there exist $R = R(X_m)$ such that $I(v) \leq 0$ for each $v \in X_m \setminus B_R(X_m)$.

Using a symmetric version of the mountain-pass theorem [20, Theorem 9.12], we conclude the proof. □

Acknowledgement

The second author was supported in part by CNPq-Brazil. The third author was supported in part by CNPq-Brazil and AGIMB-Millennium Institute-MCT/Brazil.

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